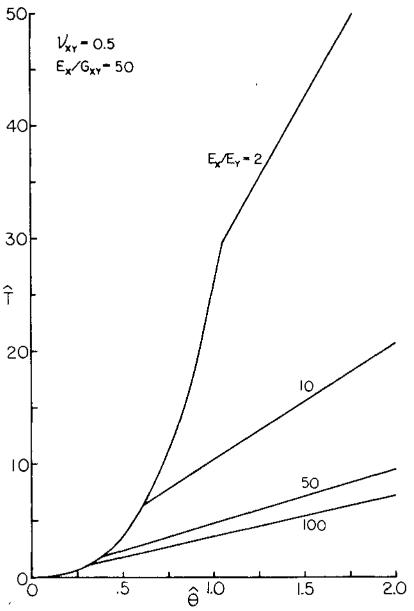

Fig. 5 Dimensionless torque as function of dimensionless twist for various values of E_x/E_y when $\nu_{xy} = 0.5$, $E_x/G_{xy} = 50$.

or

$$\hat{k}^4 + \frac{2}{(1-\nu_{xy}\nu_{yx})(E_x/E_y)}\hat{k}^2 - \frac{1}{(E_x/E_y)^4}\hat{\theta}^4 + \frac{2\nu_{xy}}{(1-\nu_{xy}\nu_{yx})(E_x/E_y)^2}\hat{\theta}^2 + \frac{1}{(1-\nu_{xy}\nu_{yx})(E_x/E_y)^2} = 0 \qquad (10)$$

In the first case

$$\hat{T} = \hat{\theta}[1+(E_x/2G_{xy})\hat{\theta}^2]$$
$$\hat{K}' = 0 \qquad (11)$$
$$\hat{N} = -\hat{\theta}^2$$

In the second case

$$\hat{k}^2 = \left|\frac{-1}{(1-\nu_{xy}\nu_{yx})(E_x/E_y)} + \frac{1}{(E_x/E_y)^{1/2}}\left(\hat{\theta}^2 - \frac{\nu_{yx}}{1-\nu_{xy}\nu_{yx}}\right)\right|$$
$$\hat{T} = \hat{\theta}\left\{1 + \frac{E_x}{2G_{xy}}\frac{1}{(1-\nu_{xy}\nu_{yx})(E_x/E_y)^{1/2}}\left[1 + \frac{\nu_{xy}}{(E_x/E_y)^{1/2}}\right]\right\} \qquad (12)$$
$$\hat{k}' = [1/(E_x/E_y)^{1/2}]\hat{k}$$
$$\hat{N} = -\frac{1+[\nu_{xy}/(E_x/E_y)^{1/2}]}{(1-\nu_{xy}\nu_{yx})(E_x/E_y)^{1/2}}$$

It is found from a comparison of strain energies that Eqs. (11) are applicable for values of $|\hat{T}|$ up to critical value $\hat{T}^*$ where

$$\hat{T}^* = \hat{\theta}^*[1+(E_x/2G_{xy})(\hat{\theta}^*)^2] \qquad (13)$$

corresponding to the following critical angle of twist

$$\hat{\theta}^* = \left[\frac{1+[\nu_{xy}/(E_x/E_y)^{1/2}]}{(1-\nu_{xy}\nu_{yx})(E_x/E_y)^{1/2}}\right]^{1/2} \qquad (14)$$

Equations (12) are applicable for values of $|\hat{T}|$ greater than $\hat{T}^*$.

Figure 2 shows variations of $\hat{\theta}^*$ as function of ν_{xy} and E_x/E_y. As E_x/E_y increases, $\hat{\theta}^*$ becomes nearly independent of ν_{xy}. Figures 3–5 show curves of $\hat{T}$ vs $\hat{\theta}$ for different values of parameters E_x/G_{xy}, ν_{xy}, and E_x/E_y. The instability phenomenon at $\hat{\theta} = \hat{\theta}^*$ is due to the fact that as the strip is twisted the midplane force becomes more resisting to the torque and as a result the strip buckles and deforms into a surface which approximates to a developable surface.

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Calculation of Mechanical Impedance by Finite Element Hybrid Model

S. T. Mau* and P. Tong†
MIT, Cambridge, Mass.

In a recent paper,[1] the mechanical impedance of damped three-layer sandwich rings (Fig. 1) are calculated using series solution. The governing differential equations are reduced to simple algebraic equations by assuming trigonometric series for the displacements. For a problem of rather simple geometry given in Ref. 1, the analytical derivation is already rather lengthy. For problems of more complex geometry, it would be very difficult to obtain solutions. It should be recognized, however, that such problems can readily be solved by the finite element method regardless of their geometry complexity. Since the problem concerns only periodically forced vibration, the frequency-dependent viscous material properties can be expressed as complex constants which are functions of the forcing frequency. The finite element system equations can be written as

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}\,e^{i\omega t} \qquad (1)$$

where **M** and **K** are system mass and stiffness matrices, and **u** and **f** are displacement force vectors, respectively. By assuming

$$\mathbf{u} = \mathbf{u}_0\,e^{i\omega t} \qquad (2)$$

Eq. (1) becomes

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{u}_0 = \mathbf{f} \qquad (3)$$

Thus the response vector $\mathbf{u}_0$ can be easily solved for any given ω. Since the material constants of the sandwich core are complex and a function of frequency, the stiffness matrix **K** is also complex and frequency dependent and the vector $\mathbf{u}_0$ is also complex. Once the complex $\mathbf{u}_0$ is obtained, the velocity $\dot{\mathbf{u}}$ can be calculated by

$$\dot{\mathbf{u}} = i\omega\mathbf{u}_0\,e^{i\omega t} \qquad (4)$$

and the mechanical impedance, which is the ratio of the amplitude of the force to the amplitude of the velocity at a point, can be calculated accordingly. The preceding solution procedure has been applied to sandwich beam problems using displacement elements.[2]

In this Note, the same procedure is applied, using a multilayer hybrid stress quadrilateral flat plate element,[3,4] to one of the sandwich ring problems solved in Ref. 1. The adaptation of complex material constants for the hybrid stress element is just as easy as that for a displacement element.

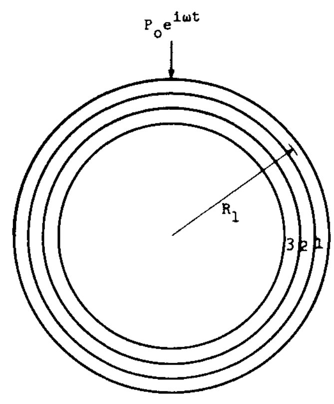

Fig. 1 Geometry and loading of the three layer sandwich ring.

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* Senior Research Engineer, Aeroelastic and Structures Research Laboratory.

† Associate Professor.

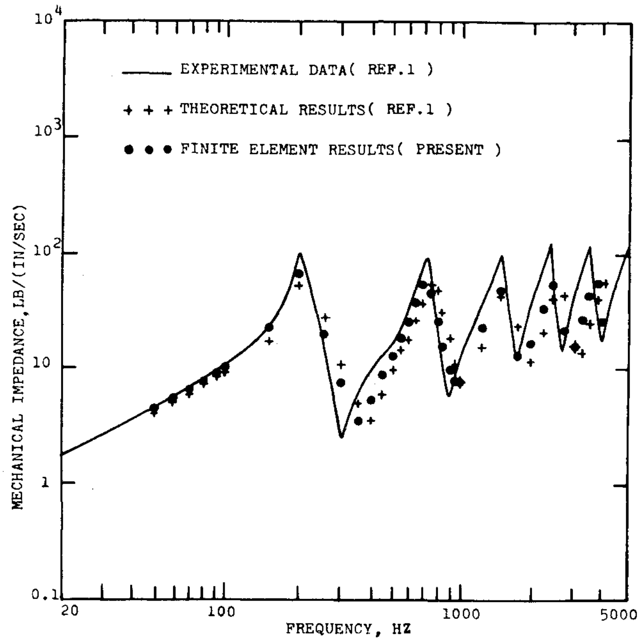

Fig. 2 Mechanical point impedance of the three layer sandwich ring under concentrated periodical loading.

The multilayer hybrid stress flat plate element has been well documented in Refs. 3 and 4. It suffices to say that the element boundary displacements are linear, that each layer has independent rotations, and that transverse shear deformation and rotatory inertia are included. The mass matrix is a nondiagonal hybrid-rational one.[5] The modelling of curved surfaces by the flat plate element has been tested and found to be satisfactory.[4]

The problem considered here is the second ring problem of Ref. 1. Numbering from the outermost layer to the innermost layer, the following data are used in the calculation: thickness (in.) $h_1 = 0.0747$, $h_2 = 0.004$, $h_3 = 0.25$; density (lb-in.^{4}/sec^2) $\gamma_1 = \gamma_3 = 0.725 \times 10^{-3}$, $\gamma_2 = 0.103 \times 10^{-3}$; Young's modulus (psi) $E_1 = E_3 = 30 \times 10^6$; Poisson's ratio $v_1 = v_3 = 0.3$. The radius to the middle surface of the first layer $R_1 = 4.3538$ in. and the width of the ring $b = 2$ in. The viscous central core is characterized by its frequency and temperature dependent complex shear modulus

$$G_2 = (1 + 1.46i)\exp[0.494 \ln(\omega/2\pi) + 3.022] \qquad (5)$$

at 72°F. In the finite element computation, the Poisson's ratio v_2 for this layer is assumed to be 0.3 and its Young's modulus E_2 is determined from G_2 and v_2.

The ring is freely supported undergoing a concentrated radial force $P_0 e^{i\omega t}$ acting on top of the first layer. Because of symmetry, only half of the ring is modelled by one row of 36 identical elements in the circumferential direction. The load P_0 is equally divided and applied to the top two nodes of the first element.

Table 1 Comparison of double and single precision solutions

Frequency, Hz ($\omega/2\pi$)	Impedance, lb/(in./sec) Double precision	Single precision
50	4.192	8.038
100	9.883	14.317
350	3.435	3.423
600	24.657	24.850
850	15.850	15.800
1500	47.914	47.886
2750	19.695	19.684
4000	25.795	25.785

The resulting mechanical impedance at the loading point is plotted in Fig. 2 against the analytical and experimental results reported in Ref. 1. It is seen that the present finite element solution is very accurate. The slightly better result over the analytical solution of Ref. 1 may be attributed to the fact that the present finite element modeling allows independent rotations for each layer and rotatory inertia is included.

For low frequency responses, some precaution must be taken to avoid getting erroneous results. The elements of the **K** matrix of this problem are in the order of 10^8 and the elements of the **M** matrix are in the order of 10^{-4}. For small ω, the effect of **M** matrix may be wiped out due to the round off of the computer and the essential rigid body response may be distorted. Such errors can be avoided simply by using double precision calculation. The effect of this can be seen in Table 1, where both double precision and single precision solutions, obtained on an IBM 370/M165, are listed for eight selected values of ω. The double precision solutions are those used in plotting Fig. 2. For $\omega/2\pi$ smaller than 350 Hz, the single precision solutions are erroneous.

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Biot's Variational Principle for Aerodynamic Ablation of Melting Solids

ANANT PRASAD*

Birla Institute of Technology and Science, Pilani, India

AND

H. C. AGRAWAL†

Indian Institute of Technology, Kanpur, India

Nomenclature

c = heat capacity per unit volume of the material
h = heat-transfer coefficient
H = heat flow vector
k = conductivity of the material
L = latent heat of the material
$q_1(t)$ = unknown surface temperature
$q_2(t)$ = melting distance
t = time

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* Lecturer, Department of Mechanical Engineering; presently, research student, Department of Mechanical Engineering, University of Liverpool, Liverpool, England.

† Associate Professor, Department of Mechanical Engineering; presently, Senior Technologist, Research & Development Laboratories, Pilkington Brothers Ltd, Lathom, Ormskirk (Lancashire), England (on leave of absence).